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The forcing edge covering number of a graph

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Abstract

An edge covering of G is a subset $S \subseteq E(G)$ such that each vertex of G is end of some edge in S. The number of edges in a minimum edge covering of G, denoted by $\beta'(G)$ is the edge covering number of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum edge covering containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing edge covering number of S, denoted by $f_{\beta'}(S)$, is the cardinality of a minimum forcing subset of S. The forcing edge covering number of G, denoted by $f_{\beta'}(G)$, is $f_{\beta'}(G) = \min\{f_{\beta'}(S)\}$, where the minimum is taken over all minimum edge coverings S in G. Some general properties satisfied by this concept is studied. The forcing edge covering number of certain classes of graphs are determined. It is shown that for every pair a, b of integer with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph G such that $f_{\beta'}(G) = a$ and $\beta'(G) = b$.

Keywords: *edge covering number, forcing edge covering number.*
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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic definitions and terminologies we refer to [1]. An edge covering of G is a subset $S \subseteq E(G)$ such that each vertex of G is end of some edge in S. The number of edges in a minimum edge covering of G, denoted by $\beta'(G)$ is the edge covering number of G. The forcing concept was first introduced and studied in minimum dominating sets in [2]. And then the forcing concept is applied in various graph parameters viz. geodetic sets, hull sets, matching's, and Steiner sets in [3, 4, 5, 6, 7] by several authors. In this paper we study the forcing concept in minimum edge covering of a connected graph. Throughout the following G denotes a connected graph with at least three vertices. The following observation is used in the sequel.

Observation 1.1. Each end-edge of G belongs to every edge covering of G.

2. The forcing edge covering number of a graph

Even though every connected graph contains a minimum edge covering, some connected graph may contain several minimum edge coverings. For each minimum edge covering S in a connected graph G, there is always some subset T of S that uniquely determines S as the minimum edge covering containing T. Such "forcing subsets" will be considered in this section.

Definition 2.1. Let G be a connected graph and S a minimum edge covering of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique minimum edge covering containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset of S*. The *forcing edge covering number* of S, denoted by $f_{\beta'}(S)$, is the cardinality of a minimum forcing subset of S. The *forcing edge covering number* of G, denoted by $f_{\beta'}(G)$, is $f_{\beta'}(G) = \min\{f_{\beta'}(S)\}$, where the minimum is taken over all minimum edge coverings S in G.

Example 2.2. For the graph G given in Figure 1, $S = \{v_1v_2, v_3v_4, v_5v_6\}$ is the unique minimum edge covering of G so that $f_{\beta'}(G) = 0$ and for the graph G given in Figure 2, $S_1 = \{v_1v_2, v_3v_4, v_3v_5\}$, $S_2 = \{v_1v_2, v_2v_5, v_3v_4,\}$ $S_3 = \{v_1v_5, v_2v_5, v_3v_4\}$, $S_4 = \{v_1v_5, v_2v_3, v_3v_4\}$ and $S_5 = \{v_1v_2, v_3v_4, v_1v_5\}$ are the only five minimum edge coverings of G such that $f_{\beta'}(S_1) = f_{\beta'}(S_4) = 1$ and $f_{\beta'}(S_2) = f_{\beta'}(S_3) = f_{\beta'}(S_5) = 2$ so that $f_{\beta'}(G) = 1$.

The next theorem follows immediately from the definition of the edge covering number and the forcing edge covering number of a connected graph G.

Theorem 2.3. For every connected graph $G, 0 \leq f_{\beta'}(G) \leq \beta'(G)$.

Remark 2.4. The bounds in Theorem 2.3 are sharp. For the graph G given in Figure 1, $f_{\beta'}(G) = 0$ and for the graph $G = K_3$, $f_{\beta'}(G) = \beta'(G) = 2$. Also, all the inequalities in the theorem are strict. For the graph G given in Figure 2, $f_{\beta'}(G) = 1$ and $\beta'(G) = 3$ so that $0 < f_{\beta'}(G) < \beta'(G)$.

In the following, we characterize graphs G for which bounds in the Theorem 2.3 attained and also graph for which $f_{\beta'}(G) = 1$.

Theorem 2.5. Let G be a connected graph. Then

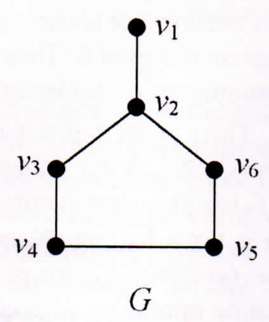

Figure 1

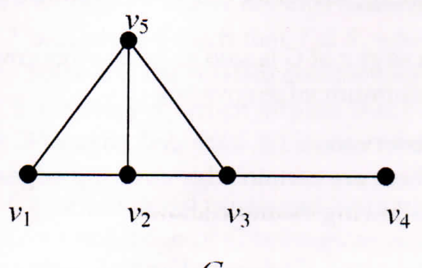

Figure 2

(a) $f_{\beta'}(G) = 0$ if and only if G has a unique minimum edge covering.

(b) $f_{\beta'}(G) = 1$ if and only if G has at least two minimum edge coverings, one of which is a unique minimum edge covering containing one of its elements, and

(c) $f_{\beta'}(G) = \beta'(G)$ if and only if no minimum edge covering of G is the unique minimum edge covering containing any of its proper subsets.

Proof.

(a) Let $f_{\beta'}(G) = 0$. Then, by definition, $f_{\beta'}(S) = 0$ for some minimum edge covering S of G so that the empty set ϕ is the minimum forcing subset for S. Since the empty set ϕ is a subset of every set, it follows that S is the unique minimum edge covering of G. The converse is clear.

(b) Let $f_{\beta'}(G) = 1$. Then by Theorem 2.5(a), G has at least two minimum edge coverings. Also, since $f_{\beta'}(G) = 1$, there is a singleton subset T of a minimum edge covering S of G such that T is not a subset of any other minimum edge coverings of G. Thus S is the unique minimum edge covering containing one of its elements. The converse is clear.

(c) Let $f_{\beta'}(G) = \beta'(G)$. Then $f_{\beta'}(S) = \beta'(G)$ for every minimum edge covering S in G. Since, $q \geq 2$, $\beta'(G) \geq 2$ and hence $f_{\beta'}(G) \geq 2$. Then by Theorem 2.5(a), G has at least two minimum edge coverings and so the empty set ϕ is not a forcing subset for any minimum edge covering of G. Since $f_{\beta'}(S) = \beta'(G)$, no proper subset of S is a forcing subset of S. Thus no minimum edge covering of G is the unique minimum edge covering containing any of its proper subsets. Conversely, the data implies that G contains more than one minimum edge covering and no subset of any minimum edge coverings S other than S is a forcing subset for S. Hence it follows that $f_{\beta'}(G) = \beta'(G)$.

Definition 2.6. An edge e of G is said to be an edge covering edge of G if e belongs to every minimum edge covering of G.

Remark 2.7. By Observation 1.1, each end edge of G is an edge covering edge of G. In fact there are certain edge covering edges which are not end edges of G as the following example shows.

Example 2.8. For the graph G given in Figure 3, $S_1 = \{v_1v_2, v_3v_6, v_4v_5\}$ and $S_2 = \{v_2v_3, v_1v_6, v_4v_5\}$ are the only two minimum edge coverings of G. It is clear that v_4v_5 is an edge covering edge of G which is not an end edge of G.

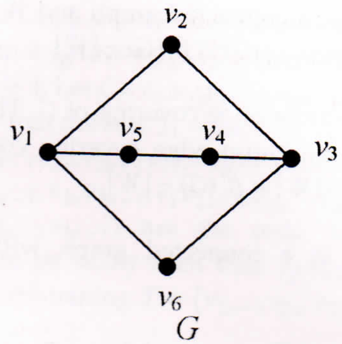

Figure 3

Theorem 2.9. Let G be a connected graph and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective minimum edge coverings in G. Then $\cap_{F\in\Im} F$ is the set of edge covering edges of G.

Proof. Let W be the set of all edge covering edges of G. We have to show that $W = \cap_{F\in\Im} F$. Let $e \in W$. Then e is an edge covering edge of G that belongs to every minimum edge covering S of G. Let $T \subseteq S$ be any minimum forcing subset for any minimum edge covering S of G. We claim that $e \notin T$. If $e \in T$, then $T' = T - \{e\}$ is a proper subset of T such that S is the unique minimum edge covering containing T' so that T' is a forcing subset for S with $|T'| < |T|$, which is a contradiction to T is a minimum forcing subset for S. Thus $e \notin T$ and so $e \in F$, where F is the relative complement of T in S. Hence $e \in \cap_{F\in\Im} F$ so that $W \subseteq \cap_{F\in\Im} F$.

Conversely, let $e \in \cap_{F\in\Im} F$. Then e belongs to the relative complement of T in S for every T and every S such that $T \subseteq S$, where T is a minimum forcing subset for S. Since F is the relative complement of T in S, we have $F \subseteq S$ and thus $e \in S$ for every S, which implies that e is an edge covering edge of G. Thus $e \in W$ and so $\cap_{F\in\Im} F \subseteq W$. Hence $W = \cap_{F\in\Im} F$.

Corollary 2.10. Let G be a connected graph and S a minimum edge covering of G. Then no edge covering edge of G belongs to any minimum forcing subset of S.

Proof. The proof is contained in the proof of the first part of Theorem 2.9.

Theorem 2.11. Let G be a connected graph and W be the set of all edge covering edges of G. Then $f_{\beta'}(G) \leq \beta'(G) - |W|$.

Proof. Let S be a minimum edge covering of G. Then $\beta'(G) = |S|, W \subseteq S$ and S is the unique minimum edge covering containing $S - W$. Thus $f_{\beta'}(G) \leq |S - W| = |S| - |W| = \beta'(G) - |W|$.

Corollary 2.12. If G is a connected graph with k end edges, then $f_{\beta'}(G) \leq \beta'(G) - k$

Proof. This follows from Observation1.1 and Theorem 2.11.

Remark 2.13. The bound in Theorem 2.11 is sharp. For the graph G given in Figure 4, $S_1 = \{v_1v_2, v_3v_4, v_5v_6, v_5v_7\}$ and $S_2 = \{v_1v_2, v_3v_4, v_5v_6, v_2v_7\}$ are the only two minimum edge coverings of G such that $f_{\beta'}(S_1) = f_{\beta'}(S_2) = 1$ and $\beta'(G) = 4$ so that $f_{\beta'}(G) = 1$. Also, $W = \{v_1v_2, v_3v_4, v_5v_6\}$ is the set of all edge covering edges of G and so $f_{\beta'}(G) = \beta'(G) - |W|$. Also, the inequality in Theorem 2.11 can be strict. For the graph G given in Figure 3, $\beta'(G) = 3$ and $f_{\beta'}(G) = 1$. Now, v_4v_5 is the only edge covering edge of G and so $f_{\beta'}(G) < \beta'(G) - |W|$.

In the following we determine the forcing edge covering number of some standard graphs.

Theorem 2.14. For any cycle $C_p (p \geq 4)$, $f_{\beta'}(C_p) = \begin{cases} 1 & \text{if } p \text{ is even} \\ 2 & \text{if } p \text{ is odd} \end{cases}$

Proof. If p is even and let $p = 2n$. Let C_p: $v_1, v_2, v_3, \ldots, v_{2n}, v_1$ be the cycle of order $2n$. Now, it is clear that the sets $S_1 = \{v_1v_2, v_3v_4, v_5v_6, \ldots, v_{2n-1}v_{2n}\}$ and

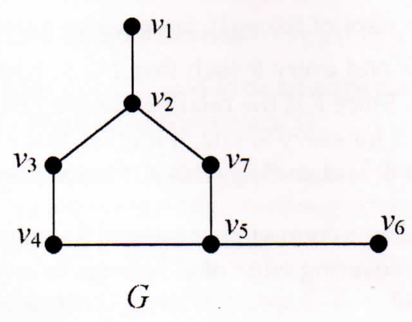

Figure 4

$S_2 = \{v_2v_3, v_4v_5, v_6v_7, \ldots, v_{2n-2}v_{2n-1}, v_{2n}v_1\}$ are the only two minimum edge coverings of C_p, such that $f_{\beta'}(S_1) = f_{\beta'}(S_2) = 1$ and so $f_{\beta'}(C_p) = 1$. Let $p \geq 5$ be odd and let $p = 2n + 1$. Let C_p: $v_1, v_2, v_3, \ldots, v_{2n+1}, v_1$ be the cycle of order $2n + 1$. Now, it is clear that the sets $S_1 = \{v_1v_2, v_3v_4, \ldots, v_{2n-1}v_{2n}, v_{2n}v_{2n+1}\}$, $S_2 = \{v_1v_2, v_3v_4, \ldots, v_{2n-1}v_{2n}, v_{2n+1}v_1\}$, $S_3 = \{v_2v_3, v_4v_5, \ldots, v_{2n}v_{2n+1}, v_{2n+1}v_1\}, \ldots, S_{2n} = \{v_{2n-2}v_{2n-1}, v_{2n}v_{2n+1}, v_1v_2, \ldots, v_{2n-3}v_{2n-2}\}$, $S_{2n+1} = \{v_{2n-1}v_{2n}, v_{2n+1}v_1, v_2v_3, \ldots, v_{2n-2}v_{2n-1}\}$ are the only $2n + 1$ minimum edge coverings of C_p. It can be easily seen that $f_{\beta'}(C_p) \geq 2$. Since S_1 is the unique edge covering containing $T = \{v_{2n-1}v_{2n}, v_{2n}v_{2n+1}\}$, it follows that $f_{\beta'}(S_1) = 2$. Thus $f_{\beta'}(C_p) = 2$.

Theorem 2.15. A set S of edges of $G = K_{n,n} (n \geq 2)$ is a minimum edge covering of G if and only if S consists of n independent edges.

Proof. Let S be any set of n independent edges of $G = K_{n,n} (n \geq 2)$. Since each vertex of G is incident with an edge of S, it follows that $\beta'(G) \leq n$. If $\beta'(G) < n$, then there exists an edge covering S' of G such that $|S'| < n$. Therefore, there exists at least one vertex v of G such that v is not incident with any edge of S' and so S' is not an edge covering of G, which is a contradiction. Hence S is a minimum edge covering of $K_{n,n}$.

Conversely, let S be a minimum edge covering of G. Let S' be any set of n independent edges of G. Then as in the first part of this theorem, S' is a minimum edge covering of G. Therefore, $|S'| = n$. Hence $|S| = n$. If S is not independent, then there exists a vertex v of G such that v is not incident with any edge of S. Hence S is not an edge covering of G, which is a contradiction. Thus S consists of n independent edges.

Theorem 2.16. For the complete bipartite graph $G = K_{n,n} (n \geq 2)$, $f_{\beta'}(G) = n - 1$.

Proof. Let $X = \{u_1, u_2, \ldots, u_n\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$ be a bipartition of G. Let S be a minimum edge covering of G such that $|S| = n$. Then by Theorem 2.15, every element of S is independent. We show that $f_{\beta'}(G) = n - 1$. Suppose that $f_{\beta'}(G) \leq n - 2$. Then there exists a forcing subset T of S such that S is the unique minimum edge covering of G containing T and $|T| \leq n - 2$. Hence there exist at least two edges $u_iv_j, u_lv_m \in S$ such that $u_iv_j, u_lv_m \notin T$ and $i \neq l, j \neq m$. Then $S_1 = S - \{u_iv_j, u_lv_m\} \cup \{u_iv_m, u_lv_j\}$ is a set of n independent edges of G containing T. By Theorem 2.15, S_1 is a minimum edge covering of, G which is a contradiction to T is a forcing subset of S. Hence $f_{\beta'}(G) = n - 1$.

Theorem 2.17. A set S of edges of $G = K_{m,n}(2 \leq m < n)$ is a minimum edge covering of G if and only if S consists of $m-1$ independent edges of G and $n-m+1$ adjacent edges of G.

Proof. Let $X = \{u_1, u_2, \ldots, u_m\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$ be a bipartition of G. Let S be any set of $m-1$ independent edges of G and $n-m+1$ adjacent edges of G. Since each vertex of G is incident with an edge of S, it follows that $\beta'(G) \leq n$. If $\beta'(G) < n$, then there exists an edge covering S' of G such that $|S'| < n$. Therefore, there exists at least one vertex v of G such that v is not incident with any edge of S' and so S' is not an edge covering of G, which is a contradiction. Hence S is a minimum edge covering of G.

Conversely, let S be a minimum edge covering of G. Let S' be any set of $m-1$ independent edges of G and $n-m+1$ adjacent edges of G. Then as in the first part of this theorem, S' is a minimum edge covering of G. Therefore, $|S'| = n$. Hence $|S| = n$. Let us assume that $S = S_1 \cup S_2$, where S_1 consists of independent edges and S_2 consists of adjacent edges of G. If $|S_1| \leq m-2$, then S_2 must contain atmost $n - |S_1|$ edges. Then there exists at least one vertex v of X such that v is not incident with any edge of S and so S is not an edge covering of G, which is a contradiction. Therefore S consists of m–1 independent edges of G and $n-m+1$ adjacent edges of G.

Theorem 2.18. For the complete bipartite graph $G = K_{m,n}(2 \leq m < n)$, $f_{\beta'}(G) = n-1$.

Proof. Let $X = \{u_1, u_2, \ldots, u_m\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$ be a bipartition of G. Let S be a minimum edge covering of G. Then by Theorem 2.17, $S = S_1 \cup S_2$, where S_1 consists of $m-1$ independent edges and S_2 consists of $n-m+1$ adjacent edges and $|S| = n$. We show that $f_{\beta'}(G) = n-1$. Suppose that $f_{\beta'}(G) \leq n-2$. Then there exists a forcing subset T of G such that S is the unique minimum edge covering of G containing T and $|T| \leq n-2$. Hence there exist at least two edges $x, y \in S$ such that $x, y \notin T$. Let us assume that $S_2 = \{u_k v_{l1}, u_k v_{l2}, \ldots, u_k v_{ln-m+1}\}$. Suppose that $x, y \in S_1$. Then $x = u_i v_j$ and $y = u_l v_m$ such that $i \neq l$ and $j \neq m$. Now, $S_3 = S - \{x, y\} \cup \{u_i v_m, u_l v_j\}$ consists of $m-1$ independent edges and $n-m+1$ adjacent edges of G containing T. By Theorem 2.17, S_3 is a minimum edge covering of G, which is a contradiction to T is a forcing subset of G. Suppose that $x, y \in S_2$. Let $x = u_k v_{l1}$ and $y = u_k v_{l2}$. Let $u_i v_j$ be an edge of S_1. Now, join the vertices $v_{l2}, v_{l3}, \ldots, v_{ln-m+1}$ to u_i. Now $S_4 = S_1 - \{u_i v_j\} \cup \{u_k v_{l1}\} \cup \{u_i v_j, u_i v_{l2}, u_i v_{l3}, \ldots, u_i v_{ln-m+1}\}$ consists of

$m-1$ independent edges and $n-m+1$ adjacent edges of G. By Theorem 2.17, S_4 is a minimum edge covering of G containing T, which is a contradiction. Suppose that $x \in S_1$ and $y \in S_2$. Let $x = u_i v_j$ and $y = u_k v_{l1}$. $S_5 = S_1 - \{u_i v_j\} \cup \{u_i v_{l1}\} \cup \{u_k v_j, u_k v_{l2}, u_k v_{l3}, \ldots, u_i v_{ln-m+1}\}$ consists of $m-1$ independent edges and $n-m+1$ adjacent edges of G containing T. By Theorem 2.17, S_5 is a minimum edge covering of G, which is a contradiction to that T is a forcing subset of G. Hence $f_{\beta'}(G) = n-1$.

Theorem 2.19. For the complete graph $G = K_p (p \geq 4)$ with p even, a set S of edges of G is a minimum edge covering of G if and only if S consists of $\frac{p}{2}$ independent edges.

Proof. The proof is similar to the proof of Theorem 2.15.

Theorem 2.20. For the complete graph $G = K_p (p \geq 4)$ with p even, $f_{\beta'}(G) = \frac{p-2}{2}$.

Proof. The proof is similar to the proof of Theorem 2.16.

Theorem 2.21. For the complete graph $G = K_p (p \geq 5)$ with p odd, a set S of edges of G is a minimum edge covering of G if and only if S consists of $\frac{p-3}{2}$ independent edges and two adjacent edges of G.

Proof. The proof is similar to the proof of Theorem 2.17.

Theorem 2.22. For the complete graph $G = K_p (p \geq 5)$ with p odd, $f_{\beta'}(G) = \frac{p-1}{2}$.

Proof. The proof is similar to the proof of Theorem 2.18.

Theorem 2.23. For the star $G = K_{1,q} (q \geq 2)$, $f_{\beta'}(G) = 0$.

Proof. For $G = K_{1,q}$, it follows from Observation 1.1 that the set of end edges of G is the unique minimum edge covering of G. Now, it follows from Theorem 2.5(a) that $f_{\beta'}(G) = 0$.

In view of Theorem 2.3, we have the following realization theorem

Theorem 2.24. For every pair a, b of integers with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph G such that $f_{\beta'}(G) = a$ and $\beta'(G) = b$.

Proof. If $a = 0$, let $G = K_{1,b}$. Then by Theorem 2.23, $f_{\beta'}(G) = 0$ and by Observation 1.1, $\beta'(G) = b$. Thus, we assume that $0 < a < b$. If $b = a+1$, then let $G = K_{b,b}$. By Theorem 2.16, $f_{\beta'}(G) = b-1 = a$ and $\beta'(G) = b$.

If $b \neq a+1$, then let $H = K_{2,a}$. Let $U = \{x,y\}$ and $W = \{w_1, w_2, \ldots, w_a\}$ be a bipartite set of H. Let G be the graph in Figure 5 obtained from H by adding $b-a$ new vertices $z, z_1, z_2, \ldots, z_{b-a-1}$ and joining each vertex $z_i (1 \leq i \leq b-a-1)$ with y and join z with x. First we show that $\beta'(G) = b$. Let $Z = \{xz, yz_1, yz_2, \ldots, yz_{b-a-1}\}$ be the set of end edges of G. By Observation 1.1, Z is a subset of every edge covering of G. It is clear that Z is not an edge covering of G. Let $H_i = \{h_i, k_i\} (1 \leq i \leq a)$, where $h_i = xw_i$ and $k_i = yw_i$. It is easily observed that every edge covering of G must contain at least one vertex from $H_i (1 \leq i \leq a)$. Thus $\beta'(G) \geq b - a + a = b$. On the other hand, since the set $S = Z \cup \{h_1, h_2, \ldots, h_a\}$ is an edge covering of G, it follows that $\beta'(G) \leq |S| = b$. Thus $\beta'(G) = b$. Next we show that $f_{\beta'}(G) = a$. Since every minimum edge covering of G contains Z, it follows from Theorem 2.11 that $f_{\beta'}(G) \leq \beta'(G) - |Z| = b - (b-a) = a$. Now, since $\beta'(G) = b$ and every edge covering of G contains S, it is easily seen that every minimum edge covering W is of the form $W \cup \{e_1, e_2, \ldots, e_a\}$, where $e_i \in H_i (1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then there exists an edge $e_j (1 \leq j \leq a)$ such that $e_j \notin T$. Let f_j be an edge of H_j distinct from e_j. Then $W_1 = (S - e_j) \cup \{f_j\}$ is a minimum edge covering properly containing T. Thus W is not the unique minimum edge covering containing T. Thus T is not a forcing subset of S. This is true for all minimum edge coverings of G and so it follows that $f_{\beta'}(G) = a$.

The upper forcing geodetic number of a graph is introduced in [8]. By the similar manner the upper forcing edge covering number of a graph is defined in the following definition.

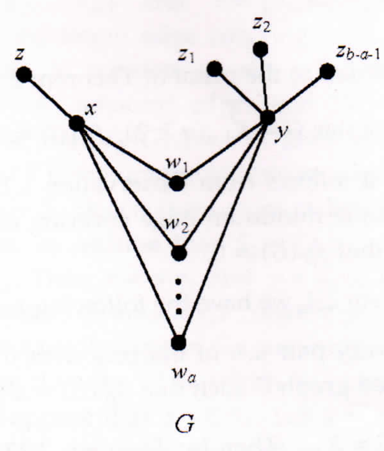

Figure 5

Definition 2.25. Let G be a connected graph and S a minimum edge covering of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique minimum edge covering containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset of* S. The *forcing edge covering number* of S, denoted by $f_{\beta'}(S)$, is the cardinality of a minimum forcing subset of S. The *upper forcing edge covering number* of G, denoted by $f^+_{\beta'}(G)$, is $f^+_{\beta'}(G) = \max\{f_{\beta'}(S)\}$, where the maximum is taken over all minimum edge coverings S in G.

For the graph G given in Figure 2, $f_{\beta'}(G) = 1$, $f^+_{\beta'}(G) = 2$ and $\beta'(G) = 3$. So we leave the following problem as open question.

Problem 2.26. For every integers a, b and c with $0 \leq a \leq b \leq c, c \geq 2$, does there exists a connected graph G such that $f_{\beta'}(G) = a$, $f^+_{\beta'}(G) = b$ and $\beta'(G) = c$.

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